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THE VALIDITY OF THE BOUSSINESQ APPROXIMATION FOR LIQUIDS AND GASES

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Abstract—A new method for obtaining approximate equations for natural convection flows is presented. The systematic application of this method leads to explicit conditions for the neglect of various terms. It is shown that this method allows the specification of the conditions under which the traditional Boussinesq approximation applies to a given Newtonian liquid or gas. The method is applied to room temperature water and air.

NOMENCLATURE

a, b, c, d, e, f, m, n, fluid property coefficients;
c_p, specific heat at constant pressure;
g, gravitational acceleration;
K, thermal conductivity;
$\mathbf{k}_i$, vertical unit vector;
L, fluid layer depth;
P, pressure;
Pr, Prandtl number;
q, velocity scale;
Ra, Rayleigh number;
T, temperature;
t, time;
$\mathbf{V}_i$, velocity vector;
$\mathbf{x}_i$, position vector.

Greek symbols

α, coefficient of thermal expansion;
β, isothermal compressibility;
Γ_{ij}, deformation rate tensor;
$\varepsilon_1, \ldots, \varepsilon_{11}$, non-dimensional parameters;
θ, temperature difference scale;
κ, thermal diffusivity;
μ, absolute viscosity;
ν, kinematic viscosity;
ρ, density;
Φ, viscous dissipation function.

Subscripts

0, denotes reference state;
s, denotes static, stably stratified atmosphere;
$\tilde{}$, denotes non-dimensional quantity.

INTRODUCTION

IN BUOYANCY driven flows, the exact governing equations are intractable. Some approximation is needed, and the simplest one which admits buoyancy is the Boussinesq approximation. This approximation is commonly understood to consist of the following:

1. Density is assumed constant except when it directly causes buoyant forces;
2. All other fluid properties are assumed constant;
3. Viscous dissipation is assumed negligible.

The first point means that the continuity equation has its incompressible form and that density is considered variable only in the gravitational term of the momentum equation. As a consequence of this assumption, acoustic phenomena cannot be treated. The other points simplify the equations so that attention is focused on the effects of buoyancy.

Although these equations are named after Boussinesq [1], they seem to have been first used by Oberbeck [2]. The plausibility argument given by Chandrasekhar [3] is often referenced, but the first attempt at a detailed derivation in a dynamical situation was made by Spiegel and Veronis [4]. They considered a perfect gas of constant properties and used an order of magnitude argument. Similar assumptions and methods were used by Gebhart [5] and Plate [6].

Mihaljan [7] used a mathematically rigorous small parameter expansion technique to derive the Boussinesq equations. He assumed that density was a linear function of temperature only and that the other properties were constant. A generalization of this approach was presented by Malkus [8, 9] who considered a perfect gas and allowed thermal diffusivity and viscosity to vary with temperature only.

The present derivation improves on previous works in a number of significant respects:

1. It applies to both liquids and gases;
2. It allows all fluid properties to vary with temperature and pressure;
3. It is mathematically straightforward;
4. It allows an explicit calculation of the region of validity of the equations.

The importance of this last point should be emphasized. The Boussinesq approximation is the basis for most of what is known about natural convection. The application of this knowledge to technological or

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environmental (USAEC [10]) problems can be meaningful only when the conditions under which the Boussinesq equations are valid are explicitly known.

The plan of the derivation can be briefly outlined. All fluid properties are assumed to vary linearly with temperature and pressure. Then the general governing equations are scaled for natural convection problems so that each non-dimensional term is of order one or less and is multiplied by a characteristic non-dimensional parameter. By requiring certain of the non-dimensional parameters to be small, the Boussinesq equations are produced. This derivation is not limited to any particular geometry. By appropriate choice of the characteristic scales, it applies to natural convection in geometries such as vertical plates and horizontal cylinders as well as to horizontal layers.

Throughout this work equations are expressed in cartesian tensor notation and the repeated index summation convention is used.

FORMULATION OF THE PROBLEM

This investigation begins with the usual forms of the continuity, momentum, and energy equations for a Newtonian fluid of variable properties and zero second viscosity (Batchelor [11]):

$$\frac{D\rho}{Dt}+\rho\frac{\partial \mathbf{V}_j}{\partial x_j}=0, \tag{1}$$

$$\rho\frac{D\mathbf{V}_i}{Dt}=-\frac{\partial P}{\partial \mathbf{x}_i}-\rho g\mathbf{k}_i+\frac{\mu\partial\Gamma_{ij}}{\partial \mathbf{x}_j}+\Gamma_{ij}\frac{\partial\mu}{\partial \mathbf{x}_j}, \tag{2}$$

$$\rho c_p\frac{DT}{Dt}=\frac{K\partial^2 T}{\partial \mathbf{x}_j^2}+\frac{\partial K}{\partial \mathbf{x}_j}\frac{\partial T}{\partial \mathbf{x}_j}+\alpha T\frac{DP}{Dt}+\mu\Phi, \tag{3}$$

where

$$\Gamma_{ij}=\frac{\partial \mathbf{V}_i}{\partial \mathbf{x}_j}+\frac{\partial \mathbf{V}_j}{\partial \mathbf{x}_i}-\tfrac{2}{3}\frac{\partial \mathbf{V}_k}{\partial \mathbf{x}_k}\delta_{ij}$$

and

$$\Phi=\tfrac{1}{2}\Gamma_{ij}\left(\frac{\partial \mathbf{V}_i}{\partial \mathbf{x}_j}+\frac{\partial \mathbf{V}_j}{\partial \mathbf{x}_i}\right).$$

In order to complete the specification of the problem, one must also have relationships for the determination of the fluid properties. Since the only reversible mode of work for a Newtonian fluid is compression, its properties are functions of two thermodynamic variables. Hence, the necessary equations may be symbolically written as

$$\rho=\rho(T,P)=\text{equation of state}, \tag{4}$$

$$c_p=c_p(T,P), \tag{5}$$

$$\mu=\mu(T,P), \tag{6}$$

$$\alpha=\alpha(T,P), \tag{7}$$

$$K=K(T,P). \tag{8}$$

Usually, these functions are imperfectly known and must be inferred from tabular data. To proceed analytically, we shall assume that each may be approximated by a linearized Taylor expansion. Eventually, we shall require that most of these properties be adequately approximated by constant values. In those instances where linear terms are retained in the final result, the legitimacy of this approximation must be verified a posteriori. There is usually no difficulty in assuming linearized Taylor expansions, but the peculiar case of water at 4C should always be kept in mind. It should also be noted that the general approach of this derivation does not hinge upon the use of linearized property variations. Other functions could be accommodated without undue distress.

The linearized approximations to equations (4)–(8) are

$$\rho=\rho_0[1-\alpha_0(T-T_0)+\beta_0(P-P_0)], \tag{9}$$

$$c_p=c_{p_0}[1+a_0(T-T_0)+b_0(P-P_0)], \tag{10}$$

$$\mu=\mu_0[1+c_0(T-T_0)+d_0(P-P_0)], \tag{11}$$

$$\alpha=\alpha_0[1+e_0(T-T_0)+f_0(P-P_0)], \tag{12}$$

$$K=K_0[1+m_0(T-T_0)+n_0(P-P_0)], \tag{13}$$

where

$$\alpha=-\frac{1}{\rho}\frac{\partial\rho}{\partial T},\qquad \beta=\frac{1}{\rho}\frac{\partial\rho}{\partial P},$$

$$a=\frac{1}{c_p}\frac{\partial c_p}{\partial T},\qquad b=\frac{1}{c_p}\frac{\partial c_p}{\partial P},$$

$$c=\frac{1}{\mu}\frac{\partial\mu}{\partial T},\qquad d=\frac{1}{\mu}\frac{\partial\mu}{\partial P},$$

$$e=\frac{1}{\alpha}\frac{\partial\alpha}{\partial T},\qquad f=\frac{1}{\alpha}\frac{\partial\alpha}{\partial P},$$

$$m=\frac{1}{K}\frac{\partial K}{\partial T},\qquad n=\frac{1}{K}\frac{\partial K}{\partial P},$$

and where the subscript 0 denotes the reference state (T_0, P_0).

Following Calder [12], we will remove the effects of static stratification. We assume a static, stably stratified atmosphere denoted by subscript s and governed by

$$\frac{\partial P_s}{\partial \mathbf{x}_i}=-\rho_s g\mathbf{k}_i \tag{14}$$

and

$$\frac{\partial}{\partial \mathbf{x}_j}\left(K_s\frac{\partial T_s}{\partial \mathbf{x}_j}\right)=0, \tag{15}$$

which are deduced from equations (2) and (3). According to equations (9) and (13), ρ_s and K_s are given by

$$\rho_s=\rho_0[1-\alpha_0(T_s-T_0)+\beta_0(P_s-P_0)] \tag{16}$$

and

$$K_s=K_0[1+m_0(T_s-T_0)+n_0(P_s-P_0)]. \tag{17}$$

The solution of equations (14)–(17) is not important at this time. Equation (14) is subtracted from the momentum equation (2), and the linearized property equations (9)–(13) are substituted into the result and into the continuity (1) and energy (3) equations. The resulting equations are

$$-\alpha_0\frac{DT}{Dt}+\beta_0\frac{DP}{Dt} + [1-\alpha_0(T-T_0)+\beta_0(P-P_0)]\frac{\partial \mathbf{V}_j}{\partial \mathbf{x}_j} = 0, \quad (18)$$

$$[1-\alpha_0(T-T_0)+\beta_0(P-P_0)]\frac{D\mathbf{V}_i}{Dt} = -\frac{1}{\rho_0}\frac{\partial(P-P_s)}{\partial \mathbf{x}_i}+\alpha_0(T-T_s)g\mathbf{k}_i-\beta_0(P-P_s)g\mathbf{k}_i + \nu_0[1+c_0(T-T_0)+d_0(P-P_0)]\frac{\partial \Gamma_{ij}}{\partial \mathbf{x}_j} + \nu_0\left[c_0\frac{\partial T}{\partial \mathbf{x}_j}+d_0\frac{\partial P}{\partial \mathbf{x}_j}\right]\Gamma_{ij}, \quad (19)$$

$$[1-\alpha_0(T-T_0)+\beta_0(P-P_0)] \times[1+a_0(T-T_0)+b_0(P-P_0)]\frac{DT}{Dt} = \kappa_0[1+m_0(T-T_0)+n_0(P-P_0)]\frac{\partial^2 T}{\partial \mathbf{x}_j^2} + \kappa_0\left[m_0\frac{\partial T}{\partial \mathbf{x}_j}+n_0\frac{\partial P}{\partial \mathbf{x}_j}\right]\frac{\partial T}{\partial \mathbf{x}_j} + \left(\frac{\alpha_0}{\rho_0 c_{p_0}}\right)[1+e_0(T-T_0)+f_0(P-P_0)] \times\Big\{(T-T_0)\frac{D}{Dt}(P-P_s)+T_0\frac{D}{Dt}(P-P_s)-(T-T_0) \times g\mathbf{V}_3\rho_0[1-\alpha_0(T_s-T_0)+\beta_0(P_s-P_0)] - T_0 g\mathbf{V}_3\rho_0[1-\alpha_0(T_s-T_0)+\beta_0(P_s-P_0)]\Big\} + \frac{\nu_0}{c_{p_0}}[1+c_0(T-T_0)+d_0(P-P_0)]\Phi, \quad (20)$$

where

$$\nu_0 = \frac{\mu_0}{\rho_0},$$

and

$$\kappa_0 = \frac{K_0}{\rho_0 c_{p_0}}.$$

In the energy equation (20), use has been made of the identity

$$T\frac{DP}{Dt} = (T-T_0)\frac{D(P-P_s)}{Dt}+\frac{T_0 D(P-P_s)}{Dt} -(T-T_0)\rho_s g\mathbf{V}_3 - T_0\rho_s g\mathbf{V}_3$$

and of equation (16) for reasons of later convenience.

NON-DIMENSIONALIZATION OF THE EQUATIONS

In order to recover the Boussinesq equations from equations (18)–(20), many terms must be eliminated. The neglect of these terms is justified in the following manner. The equations are non-dimensionalized with suitable scales so that all functions of the non-dimensional field variables may be considered to be of order one or less. The relative importance of these terms is then given by non-dimensional constant multipliers. By requiring that the multipliers be small, the terms in question may be neglected.

The choice of scales depends on the particular problem in question. For the sake of definiteness, we shall consider natural convection in a horizontal fluid layer of vertical thickness L across which a temperature difference θ is maintained, i.e. the Rayleigh–Bénard problem. Lengths are non-dimensionalized by the distance L. Temperature differences of the forms $(T-T_0)$ and (T_s-T_0) as well as temperatures acted upon by derivatives are of the same order and may be properly scaled by θ. A meaningful velocity scale must be related to the intensity of motion. For this reason, scales related to the molecular diffusivities such as ν_0/L (Kreith [13], Giorgini and Travis [14]) and κ_0/L (Mihaljan [7]) are unrealistic. The scale $q=\sqrt{(\alpha_0 g\theta L)}$ is used here (Malkus [8, 9]). This may be thought of as the "free fall" velocity of a thermal. The dynamic pressure $(P-P_s)$ is scaled by ρq^2, while differences in pressure between different elevations, $(P-P_0)$ and (P_s-P_0), and derivatives of pressure are scaled by $\rho_0 gL$. With these remarks in mind, the non-dimensional variables (denoted by tildes) are seen to be

$$\left.\begin{aligned} \mathbf{x}_i &= L\tilde{\mathbf{x}}_i,\\ T-T_0 &= \theta(\tilde{T}-\tilde{T}_0),\\ \mathbf{V}_i &= q\tilde{\mathbf{V}}_i = (\alpha_0 g\theta L)^{\frac{1}{2}}\tilde{\mathbf{V}}_i,\\ t &= \frac{L}{q}\tilde{t} = \left(\frac{L}{\alpha_0 g\theta}\right)^{\frac{1}{2}}\tilde{t},\\ P-P_s &= \rho q^2(\tilde{P}-\tilde{P}_s) = \rho_0\alpha_0 g\theta L(\tilde{P}-\tilde{P}_s),\\ P-P_0 &= \rho_0 gL(\tilde{P}-\tilde{P}_0),\\ \Gamma_{ij} &= \frac{q}{L}\tilde{\Gamma}_{ij} = \left(\frac{\alpha_0 g\theta}{L}\right)^{\frac{1}{2}}\tilde{\Gamma}_{ij},\\ \Phi &= \frac{q^2}{L^2}\tilde{\Phi} = \frac{\alpha_0 g\theta}{L}\tilde{\Phi}. \end{aligned}\right\} \quad (21)$$

With these definitions, equations (18)–(20) may be rearranged to read

$$-\varepsilon_1\frac{D\tilde{T}}{D\tilde{t}}+\varepsilon_2\frac{D\tilde{P}}{D\tilde{t}} + [1-\varepsilon_1(\tilde{T}-\tilde{T}_0)+\varepsilon_2(\tilde{P}-\tilde{P}_0)]\frac{\partial \tilde{\mathbf{V}}_j}{\partial \tilde{\mathbf{x}}_j} = 0, \quad (22)$$

$$\left.\begin{aligned} &[1-\varepsilon_1(\tilde{T}-\tilde{T}_0)+\varepsilon_2(\tilde{P}-\tilde{P}_0)]\frac{D\tilde{\mathbf{V}}_i}{D\tilde{t}}\\ &= -\frac{\partial(\tilde{P}-\tilde{P}_s)}{\partial \tilde{\mathbf{x}}_i}+(\tilde{T}-\tilde{T}_s)\mathbf{k}_i-\varepsilon_2(\tilde{P}-\tilde{P}_s)\mathbf{k}_i\\ &\quad+\left(\frac{Pr}{Ra}\right)^{\frac{1}{2}}[1+\varepsilon_3(\tilde{T}-\tilde{T}_0)+\varepsilon_4(\tilde{P}-\tilde{P}_0)]\frac{\partial \tilde{\Gamma}_{ij}}{\partial \tilde{\mathbf{x}}_j}\\ &\quad+\left(\frac{Pr}{Ra}\right)^{\frac{1}{2}}\left[\varepsilon_3\frac{\partial \tilde{T}}{\partial \tilde{\mathbf{x}}_j}+\varepsilon_4\frac{\partial \tilde{P}}{\partial \tilde{\mathbf{x}}_j}\right]\tilde{\Gamma}_{ij}, \end{aligned}\right\} \quad (23)$$

$$\left.\begin{aligned}
&[1-\varepsilon_1(\tilde{T}-\tilde{T}_0)+\varepsilon_2(\tilde{P}-\tilde{P}_0)] \\
&\quad\times[1+\varepsilon_5(\tilde{T}-\tilde{T}_0)+\varepsilon_6(\tilde{P}-\tilde{P}_0)]\frac{D\tilde{T}}{D\tilde{t}} \\
&=\frac{1}{(PrRa)^{\frac{1}{2}}}[1+\varepsilon_7(\tilde{T}-\tilde{T}_0)+\varepsilon_8(\tilde{P}-\tilde{P}_0)]\frac{\partial^2\tilde{T}}{\partial\tilde{\mathbf{x}}_j^2} \\
&\quad+\frac{1}{(PrRa)^{\frac{1}{2}}}\left[\varepsilon_7\frac{\partial\tilde{T}}{\partial\tilde{\mathbf{x}}_j}+\varepsilon_8\frac{\partial\tilde{P}}{\partial\tilde{\mathbf{x}}_j}\right]\frac{\partial\tilde{T}}{\partial\tilde{\mathbf{x}}_j} \\
&\quad+\varepsilon_{11}[1+\varepsilon_9(\tilde{T}-\tilde{T}_0)+\varepsilon_{10}(\tilde{P}-\tilde{P}_0)] \\
&\quad\times\left\{\varepsilon_1(\tilde{T}-\tilde{T}_0)\frac{D}{D\tilde{t}}(\tilde{P}-\tilde{P}_s)+\varepsilon_1\left(\frac{T_0}{\theta}\right)\frac{D(\tilde{P}-\tilde{P}_s)}{D\tilde{t}}\right. \\
&\quad-(\tilde{T}-\tilde{T}_0)\tilde{\mathbf{V}}_3[1-\varepsilon_1(\tilde{T}_s-\tilde{T}_0)+\varepsilon_2(\tilde{P}_s-\tilde{P}_0)] \\
&\quad\left.-\left(\frac{T_0}{\theta}\right)\tilde{\mathbf{V}}_3[1-\varepsilon_1(\tilde{T}_s-\tilde{T}_0)+\varepsilon_2(\tilde{P}_s-\tilde{P}_0)]\right\} \\
&\quad+\varepsilon_{11}\left(\frac{Pr}{Ra}\right)^{\frac{1}{2}}[1+\varepsilon_3(\tilde{T}-\tilde{T}_0)+\varepsilon_4(\tilde{P}-\tilde{P}_0)]\tilde{\Phi},
\end{aligned}\right\} \quad (24)$$

where

$$\left.\begin{aligned}
&\varepsilon_1=\alpha_0\theta, && \varepsilon_2=\beta_0\rho_0 gL, \\
&\varepsilon_3=c_0\theta, && \varepsilon_4=d_0\rho_0 gL, \\
&\varepsilon_5=a_0\theta, && \varepsilon_6=b_0\rho_0 gL, \\
&\varepsilon_7=m_0\theta, && \varepsilon_8=n_0\rho_0 gL, \\
&\varepsilon_9=e_0\theta, && \varepsilon_{10}=f_0\rho_0 gL, \\
& && \varepsilon_{11}=\frac{\alpha_0 gL}{c_{p_0}},
\end{aligned}\right\} \quad (25)$$

$$Ra=\frac{\alpha_0 g\theta L^3}{\kappa_0\nu_0},$$

and

$$Pr=\frac{\nu_0}{\kappa_0}.$$

APPROXIMATION OF THE EQUATIONS

By requiring that all the ε factors be small so that the terms they multiply are small, one can obtain the Boussinesq equations from equations (22)–(24). One must recognize, however, that the viscous term in the momentum equation and the conduction term in the energy equation are also multiplied by parameters that are frequently small. Nevertheless, they must be retained when boundaries are present. Thus, we will approach the task of simplifying equations (22)–(24) in a stepwise fashion.

It is known from many convection problems that the effects of property variations are usually rather uninteresting. Besides, specifying property functions is often very difficult. Thus, our first step is to justify use of constant values for ρ, c_p, μ, α and K when these properties appear as multipliers in equations (22)–(24). Inspection shows that this requires that $\varepsilon_1,\ldots,\varepsilon_{10}$ be small. Choosing T_0 as the average of the upper and lower boundary temperatures and P_0 as the hydrostatic pressure at the mid-level, the values of $(\tilde{T}-\tilde{T}_0)$, $(\tilde{T}_s-T_0)$, $(\tilde{P}-\tilde{P}_0)$, and $(\tilde{P}_s-\tilde{P}_0)$ are at most ~ 0.5. Hence requiring

$$|\varepsilon_1|,\ldots,|\varepsilon_{10}|\leqslant 0.1 \quad (26)$$

can lead to an error of at most 10% in terms of the form $[1+\varepsilon_i(\tilde{T}-\tilde{T}_0)+\varepsilon_j(\tilde{P}-\tilde{P}_0)]$. We shall assume that such a condition is adequate to approximate these expressions with 1. Hence, equations (22)–(24) become

$$-\varepsilon_1\frac{D\tilde{T}}{D\tilde{t}}+\varepsilon_2\frac{D\tilde{P}}{D\tilde{t}}+\frac{\partial\tilde{\mathbf{V}}_j}{\partial\tilde{\mathbf{x}}_j}=0, \quad (27)$$

$$\left.\begin{aligned}
\frac{D\tilde{\mathbf{V}}_i}{D\tilde{t}}&=-\frac{\partial(\tilde{P}-\tilde{P}_s)}{\partial\tilde{\mathbf{x}}_i}+(\tilde{T}-\tilde{T}_s)\mathbf{k}_i-\varepsilon_2(\tilde{P}-\tilde{P}_s)\mathbf{k}_i \\
&\quad+\left(\frac{Pr}{Ra}\right)^{\frac{1}{2}}\left[\frac{\partial\tilde{\Gamma}_{ij}}{\partial\tilde{\mathbf{x}}_j}+\left(\varepsilon_3\frac{\partial\tilde{T}}{\partial\tilde{\mathbf{x}}_j}+\varepsilon_4\frac{\partial\tilde{P}}{\partial\tilde{\mathbf{x}}_j}\right)\tilde{\Gamma}_{ij}\right],
\end{aligned}\right\} \quad (28)$$

$$\left.\begin{aligned}
\frac{D\tilde{T}}{D\tilde{t}}&=\frac{1}{(PrRa)^{\frac{1}{2}}}\left[\frac{\partial^2\tilde{T}}{\partial\tilde{\mathbf{x}}_j^2}+\left(\varepsilon_7\frac{\partial\tilde{T}}{\partial\tilde{\mathbf{x}}_j}+\varepsilon_8\frac{\partial\tilde{P}}{\partial\tilde{\mathbf{x}}_j}\right)\frac{\partial\tilde{T}}{\partial\tilde{\mathbf{x}}_j}\right] \\
&\quad+\varepsilon_{11}\left[\varepsilon_1(\tilde{T}-\tilde{T}_0)\frac{D}{D\tilde{t}}(\tilde{P}-\tilde{P}_s)+\varepsilon_1\left(\frac{T_0}{\theta}\right)\right. \\
&\quad\left.\times\frac{D(\tilde{P}-\tilde{P}_s)}{D\tilde{t}}-(\tilde{T}-\tilde{T}_0)\tilde{\mathbf{V}}_3-\left(\frac{T_0}{\theta}\right)\tilde{\mathbf{V}}_3\right] \\
&\quad+\varepsilon_{11}\left(\frac{Pr}{Ra}\right)^{\frac{1}{2}}\tilde{\Phi}.
\end{aligned}\right\} \quad (29)$$

This first approximation is equivalent to using subscript zero values for all properties (except density in the buoyancy term) when they appear as multipliers, but retaining the linear approximations in the derivatives.

Without introducing any new assumptions it is possible to neglect additional terms in equations (27)–(29), since those multiplied by $\varepsilon_1,\ldots,\varepsilon_{10}$ are smaller than terms not containing these factors.

$$\frac{\partial\tilde{\mathbf{V}}_j}{\partial\tilde{\mathbf{x}}_j}=0, \quad (30)$$

$$\frac{D\tilde{\mathbf{V}}_i}{D\tilde{t}}=-\frac{\partial(\tilde{P}-\tilde{P}_s)}{\partial\tilde{\mathbf{x}}_i}+(\tilde{T}-\tilde{T}_s)\mathbf{k}_i+\left(\frac{Pr}{Ra}\right)^{\frac{1}{2}}\frac{\partial\tilde{\Gamma}_{ij}}{\partial\tilde{\mathbf{x}}_j}, \quad (31)$$

$$\frac{D\tilde{T}}{D\tilde{t}}=\frac{1}{(PrRa)^{\frac{1}{2}}}\frac{\partial^2\tilde{T}}{\partial\tilde{\mathbf{x}}_j^2}-\varepsilon_{11}\left(\frac{T_0}{\theta}\right)\tilde{\mathbf{V}}_3+\varepsilon_{11}\left(\frac{Pr}{Ra}\right)^{\frac{1}{2}}\tilde{\Phi}. \quad (32)$$

In equation (32) we have assumed that $T_0/\theta \gtrsim 10$ and retained only the largest of the pressure work terms. Notice that equations (30)–(32) still contain terms representing every distinct physical mechanism which was present in the original set [equations (1)–(3)]. Equations (30)–(32) will be called the extended Boussinesq equations.

To proceed to the next order of approximation, we shall neglect in equation (32) the pressure work term if its multiplier is small with respect to 1 and the viscous dissipation term if it is much less than the conduction term (since both are "boundary layer" type terms). These conditions imply

$$\left|\varepsilon_{11}\left(\frac{T_0}{\theta}\right)\right|\leqslant 0.1 \quad (33)$$

and

$$|\varepsilon_{11}|\left(\frac{Pr}{Ra}\right)^{\frac{1}{2}}\leqslant 0.1\frac{1}{(PrRa)^{\frac{1}{2}}}, \quad (34)$$

respectively.

If the problem in question satisfies conditions (26), (33), and (34), it is governed by the strict Boussinesq equations. In dimensional form these are

$$\frac{\partial \mathbf{V}_j}{\partial \mathbf{x}_i} = 0, \tag{35}$$

$$\frac{D\mathbf{V}_i}{Dt} = -\frac{1}{\rho_0}\frac{\partial (P-P_s)}{\partial \mathbf{x}_i} + \alpha_0\, \mathbf{g}(T-T_s)\mathbf{k}_i + \nu_0 \frac{\partial^2 \mathbf{V}_i}{\partial \mathbf{x}_j^2}, \tag{36}$$

$$\frac{DT}{Dt} = \kappa_0 \frac{\partial^2 T}{\partial \mathbf{x}_j^2}. \tag{37}$$

If either equation (33) or (34) is violated, the corresponding term must be retained.

RANGE OF THE APPROXIMATE EQUATIONS

As concrete examples of the method just presented, consider the cases of water and air at $T_0 = 15\,\mathrm{C}$ and $P_0 = 1$ atm. Table 1 lists the required constants, obtained from Batchelor [11] unless otherwise noted. In Table 2 the ε parameters are evaluated in terms of θ and L. For water, the most demanding of the requirements (26) are that ε_9 (temperature variation of α) and ε_8 (pressure variation of K) be small. These require that

$$\theta \leqslant 1.25\,\mathrm{C} \tag{38}$$

and

$$L \leqslant 2.4 \times 10^5\ \mathrm{cm}. \tag{39}$$

Requirements (33) and (34) may be rewritten as

$$\frac{L}{\theta} \leqslant \frac{0.1\, c_{p_0}}{\alpha_0 g T_0} = 9.9 \times 10^4\ \mathrm{cm/C} \tag{40}$$

and

$$L \leqslant \frac{0.1\, c_{p_0}}{\alpha_0 g Pr} = 3.5 \times 10^6\ \mathrm{cm}. \tag{41}$$

In the case of water (41) is less demanding than (39). Figure 1 shows the restrictions due to conditions (38)–(40). It is seen that the strict Boussinesq equations are valid in the shaded region to a maximum Rayleigh number of more than 10^{19}. This is about 14 decades above the onset of turbulence in this geometry (Krishnamurti [15]). As far as the extended Boussinesq equations are concerned, since (41) is more relaxed than (39), they can be written without the dissipation term and their validity extends throughout the lower rectangular region of Fig. 1.

Table 1. Fluid properties for water and air at $T_0 = 15\,\mathrm{C}$, $P_0 = 1$ atm

	Water	Air
ρ_0	$1\ \mathrm{g/cm^3}$	$1.2 \times 10^{-3}\ \mathrm{g/cm^3}$
c_{p_0}	4.2×10^7 ergs/gm C	10^7 ergs/g C
ν_0	$1.1 \times 10^{-2}\ \mathrm{cm^2/s}$	$0.145\ \mathrm{cm^2/s}$
Pr	8.1	0.72
α_0	$1.5 \times 10^{-4}\ \mathrm{C^{-1}}$	$3.5 \times 10^{-3}\ \mathrm{C^{-1}}$
β_0	$4.9 \times 10^{-11}\ \mathrm{cm^2/dyn}$	$10^{-6}\ \mathrm{cm^2/dyn}$
a_0	$-2.4 \times 10^{-4}\ \mathrm{C^{-1}}$	$4.5 \times 10^{-5}\ \mathrm{C^{-1}}$†
b_0	$-2.5 \times 10^{-10}\ \mathrm{cm^2/dyn}$*	$1.9 \times 10^{-9}\ \mathrm{cm^2/dyn}$‡
c_0	$-2.7 \times 10^{-2}\ \mathrm{C^{-1}}$	$2.8 \times 10^{-3}\ \mathrm{C^{-1}}$
d_0	$-2.7 \times 10^{-11}\ \mathrm{cm^2/dyn}$*	0 (kinetic theory)
e_0	$8.0 \times 10^{-2}\ \mathrm{C^{-1}}$	$-3.6 \times 10^{-3}\ \mathrm{C^{-1}}$ (perfect gas)
f_0	0 (no data)	0 (perfect gas)
m_0	$1.7 \times 10^{-3}\ \mathrm{C^{-1}}$	$2.4 \times 10^{-3}\ \mathrm{C^{-1}}$
n_0	$4.3 \times 10^{-10}\ \mathrm{cm^2/dyn}$*	0 (kinetic theory)

*Meyer *et al.* [16].
†Kreith [13].
‡Hilsenrath *et al.* [17].

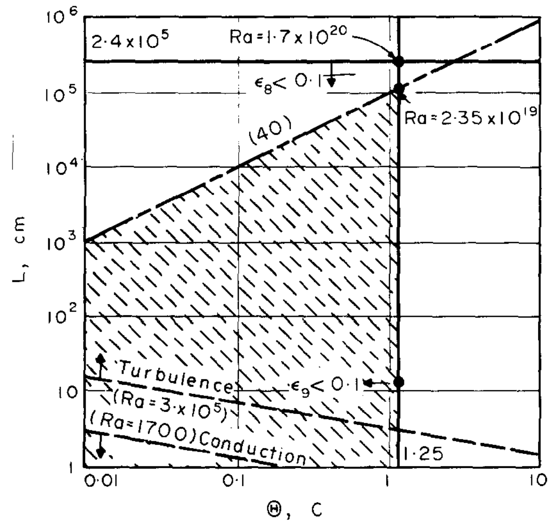

FIG. 1. Regions of validity of the Boussinesq approximation in water ($T_0 = 15\,\mathrm{C}$, $P_0 = 1$ atm).

In the case of air, similar results are obtained. The most demanding of (26) are that ε_1 (variation of ρ with T) and ε_2 (variation of ρ with P) be small. These require that

$$\theta \leqslant 28.6\,\mathrm{C} \tag{42}$$

and

$$L \leqslant 8.3 \times 10^4\ \mathrm{cm}. \tag{43}$$

Conditions (33) and (34) are

$$\frac{L}{\theta} < 1020\ \mathrm{cm/C} \tag{44}$$

and

$$L < 4.1 \times 10^5\ \mathrm{cm}. \tag{45}$$

Table 2. Non-dimensional parameters (25) for water and air at $T_0 = 15\,\mathrm{C}$, $P_0 = 1$ atm

	Water	Air
ε_1	$1.5 \times 10^{-4}\,\theta$	$3.5 \times 10^{-3}\,\theta$
ε_2	$4.8 \times 10^{-8}\,L$	$1.2 \times 10^{-6}\,L$
ε_3	$-2.7 \times 10^{-2}\,\theta$	$2.8 \times 10^{-3}\,\theta$
ε_4	$-2.7 \times 10^{-8}\,L$	0
ε_5	$-2.4 \times 10^{-4}\,\theta$	$4.5 \times 10^{-5}\,\theta$
ε_6	$-2.4 \times 10^{-7}\,L$	$2.3 \times 10^{-9}\,L$
ε_7	$1.7 \times 10^{-3}\,\theta$	$2.4 \times 10^{-3}\,\theta$
ε_8	$4.2 \times 10^{-7}\,L$	0
ε_9	$8.0 \times 10^{-2}\,\theta$	$-3.6 \times 10^{-3}\,\theta$
ε_{10}	0	0
ε_{11}	$3.5 \times 10^{-9}\,L$	$3.6 \times 10^{-7}\,L$

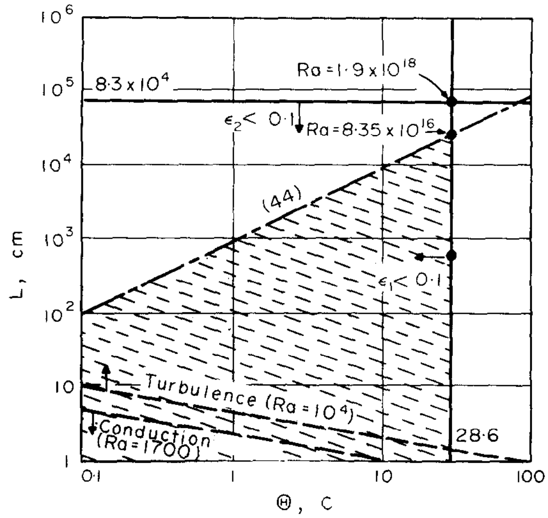

FIG. 2. Regions of validity of the Boussinesq approximation in air ($T_0 = 15$ C, $P_0 = 1$ atm).

Once again (45) is automatically satisfied. The requirements of conditions (42)–(44) are illustrated in Fig. 2. The strict Boussinesq equations may be applied in the shaded region up to a maximum Rayleigh number of nearly 10^{17}, almost 13 decades above the onset of turbulence (Krishnamurti [15]). The extended Boussinesq equations without dissipation are valid in the rectangular region.

CONCLUSIONS

The strict Boussinesq equations (35)–(37) are the basis for nearly all analyses of natural convection. The derivation presented here is the first which is simultaneously valid for liquids and gases. It is also the first derivation to be logically consistent with the state postulate of thermodynamics by allowing all properties to be functions of two state variables. These achievements, while aesthetically satisfying, do not have great practical merit *per se*. There are, nevertheless, two very important practical advantages of the derivation presented above. In the first place, this derivation makes the limits of validity of the equations explicit. Second, the method used here allows deviations from strict Boussinesq behavior to be accounted for in a systematic manner. By means of the method introduced in this paper, the conditions under which the Boussinesq approximation can be applied can be determined for any given Newtonian fluid and reference condition.

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SUR LA VALIDITE DE L'APPROXIMATION DE BOUSSINESQ POUR LES LIQUIDES ET LES GAZ

Résumé—On présente une nouvelle méthode d'obtention des équations approchées des écoulements en convection naturelle. L'application systématique de cette méthode conduit à des conditions précises sur les divers termes à négliger. On montre que cette méthode permet de spécifier les conditions sous lesquelles s'applique l'approximation habituelle de Boussinesq dans un liquide ou un gaz Newtonien. La méthode est appliquée à l'eau et à l'air à température ambiante.

DIE GÜLTIGKEIT DER BOUSSINESQ-NÄHERUNG FÜR FLÜSSIGKEITEN UND GASE

Zusammenfassung—Um Näherungsgleichungen für freie Konvektionsströmungen zu erhalten, wird eine neue Methode dargestellt. Die systematische Anwendung dieser Methode führt zu expliziten Bedingungen für die Vernachlässigung verschiedener Terme. Es wird gezeigt, wie nach dieser Methode die Bedingungen dargestellt werden können, für die die traditionelle Boussinesq-Näherung auf eine gegebene Newtonsche Flüssigkeit oder ein Gas angewendet werden kann. Die Methode wird für Wasser und Luft bei Raumtemperatur angewandt.

СПРАВЕДЛИВОСТЬ ПРИМЕНЕНИЯ ПРИБЛИЖЕНИЯ БУССЕНЕСКА ДЛЯ ЖИДКОСТЕЙ И ГАЗОВ

Аннотация — Приводится новый метод вывода приближенных уравнений для задач естественной конвекции. Использование данного метода позволяет в явном виде выяснить условия, при которых можно пренебречь некоторыми членами в уравнениях. Показано, что этот метод определяет условия, при которых традиционное приближение Буссенеска может быть использовано для случая ньютоновской жидкости или газа. Метод используется для воды и воздуха при комнатной температуре.